A NOTE ON WEAK CONVERGENCE OF SINGULAR INTEGRALS IN METRIC SPACES

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ABSTRACT. We prove that in any metric space (X, d) the singular integral operators

$$T^k_{\mu,\varepsilon}(f)(x) = \int_{X \setminus B(x,\varepsilon)} k(x,y) f(y) d\mu(y).$$

converge weakly in some dense subspaces of $L^2(\mu)$ under minimal regularity assumptions for the measures and the kernels.

1. Introduction

A Radon measure on a metric space (X, d) has s-growth if there exists some constant c_μ such that $\mu(B(x,r)) \leq c_\mu r^s$ for all $x \in X$, $r > 0$.

We say that $k(\cdot,\cdot) : X \times X \setminus \{(x,y) \in X \times X : x = y\} \to \mathbb{R}$ is an s-dimensional kernel if there exists a constant $c > 0$ such that for all $x, y \in X$, $x \neq y$:

$$|k(x,y)| \leq c\, d(x,y)^{-s}.$$

The kernel k is antisymmetric if $k(x,y) = -k(y,x)$ for all distinct $x, y \in X$.

Given a positive Radon measure ν on X and an s-dimensional kernel k, we define

$$T^k\nu(x) := \int k(x,y)\, d\nu(y), \qquad x \in X \setminus \operatorname{spt}\nu.$$

This integral may not converge when $x \in \operatorname{spt}\nu$. For this reason, we consider the following ε-truncated operators T^k_ε, $\varepsilon > 0$:

$$T^k_\varepsilon \nu(x) := \int_{d(x,y)>\varepsilon} k(x,y)\, d\nu(y), \qquad x \in X.$$

Given a fixed positive Radon measure μ on X and $f \in L^1_{\mathrm{loc}}(\mu)$, we write

$$T^k_\mu f(x) := T^k(f\,\mu)(x), \qquad x \in X \setminus \operatorname{spt}(f\,\mu),$$

and

$$T^k_{\mu,\varepsilon} f(x) := T^k_\varepsilon(f\,\mu)(x).$$

Concerning the limit properties of the operators $T^k_{\mu,\varepsilon}$ one can ask if the limit, the so called principal value of T,

$$\lim_{\varepsilon \to 0} T^k_{\mu,\varepsilon}(f)(x),$$

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exists μ almost everywhere. When μ is the Lebesgue measure in $\mathbb{R}^d$, and k is a standard Calderón-Zygmund kernel, due to cancellations and the denseness of smooth functions in L^1, the principal values exist almost everywhere for L^1-functions. For more general measures, the question is more complicated. Let n be an integer, $0 < n < d$, and consider the coordinate Riesz kernels

$$R_i^n(x) = \frac{x_i}{|x|^{n+1}} \text{ for } i = 1, \dots, d.$$

Tolsa proved in [T] that if $E \subset \mathbb{R}^d$ has finite n-dimensional Hausdorff measure $\mathcal{H}^n$ the principal values

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(x,\varepsilon)} \frac{x_i - y_i}{|x-y|^{m+1}} d\mathcal{H}^n(y)$$

exist $\mathcal{H}^n$ almost everywhere in E if and only if the set E is n-rectifiable i.e. if there exist n-dimensional Lipschitz surfaces M_i, $i \in \mathbb{N}$, such that

$$\mathcal{H}^n(E \setminus \cup_{i=1}^{\infty} M_i) = 0.$$

Mattila and Preiss had obtained the same result earlier, in [MP] under some stronger assumptions for the set E. It becomes obvious that the existence of principal values is deeply related to the geometry of the set E.

Assuming $L^2(\mu)$-boundedness for the operators T_μ^k one could have expected that more could be deduced about the structure of μ and the existence of principal values, but this is a hard and, in a large extent, open problem. Dating from 1991 the David-Semmes conjecture, see [DS], asks if the $L^2(\mu)$-boundedness of the operators associated with the n-dimensional Riesz kernels suffices to imply n-uniform rectifiabilty, which can be thought as a quantitative version of rectifiability. In the very recent deep work [NToV], Nazarov, Tolsa and Volberg resolved the conjecture in the codimension 1 case, that is for $n = d-1$. Mattila, Melnikov and Verdera in [MMV], using a special symmetrization property of the Cauchy kernel, had earlier proved the conjecture in the case of 1-dimensional Riesz kernels. For all other dimensions and for other kernels few things are known. In fact, there are several examples of kernels whose boundedness does not imply rectifiability, see [C], [D] and [H]. For some recent positive results involving other kernels see [CMPT].

Let μ be a finite Radon measure and let k be an antisymmetric kernel in a complete metric space (X, d) where the Vitali covering theorem holds for μ and the family of closed balls defined by d. Mattila and Verdera in [MV] showed that in this case the $L^2(\mu)$-boundedness of the operators $T_{\mu,\varepsilon}^k$ forces them to converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T_\mu^k : L^2(\mu) \to L^2(\mu)$ such that for all $f, g \in L^2(\mu)$,

$$\lim_{\varepsilon \to 0} \int T_{\mu,\varepsilon}^k(f)(x) g(x) d\mu(x) = \int T_\mu^k(f)(x) g(x) d\mu(x).$$

Furthermore notions of weak convergence have been recently used by Nazarov, Tolsa and Volberg in [NToV].

Motivated by these developments it is natural to ask if limits of this type might exist if we remove the very strong L^2-boundedness assumption. We prove that the operators $T^k_{\mu,\varepsilon}$ converge weakly in dense subspaces of $L^2(\mu)$ under minimal assumptions for the measures and the kernels in general metric spaces. Denote by $\mathcal{X}_B$ the space of all finite linear combinations of characteristic functions of balls in X,

$$\mathcal{X}_B = \left\{ \sum_{i=1}^n a_i \chi_{B(z_i,r_i)} : n \in \mathbb{N},\, a_i \in \mathbb{R},\, z_i \in X,\, r_i > 0 \right\}.$$

Whenever Vitali's covering theorem holds for the closed balls in (X, d) the space $\mathcal{X}_B$ is dense in $L^2(\mu)$. When $X = \mathbb{R}^d$ Vitali's covering theorem holds for any Radon measure μ and the closed balls defined by various metrics (including the standard d_p metrics for $1 \leq p \leq \infty$) as a consequence of Besicovitch's covering theorem, see [M, Theorem 2.8]. Furthermore Vitali's covering theorem holds for any metric space (X, d) whenever μ is doubling, that is when there exists some constant C such that for all balls B, $\mu(2B) \leq C\mu(B)$, see [F, Section 2.8].

Theorem 1.1. *Let μ be a finite Radon measure with s-growth and k an antisymmetric s-dimensional kernel on a metric space (X, d). If the Vitali Covering theorem holds for the closed balls in (X, d) then there exists subsets $\mathcal{X}'_B \subset \mathcal{X}_B$ which are dense in $L^2(\mu)$ and the weak limits*

$$\lim_{\varepsilon \to 0} \int T^k_{\mu,\varepsilon} f(x)\, g(x) d\mu(x)$$

exist for all $f, g \in \mathcal{X}'_B$.

Until now Theorem 1.1 was only known for measures with $(d-1)$-growth in $\mathbb{R}^d$ under some smoothness assumptions for the kernels, see [CM]. We thus extend the result from [CM] to measures with s-growth for arbitrary s in metric spaces where Vitali's covering theorem holds for the family of closed balls without requiring any smoothness for the kernels. Our proof follows a completely different strategy using an "exponential growth" lemma for probability measures on intervals and is self contained (unlike the proof from [CM] which depends on several $L^2(\nu)$ to $L^2(\mu)$ boundedness results for separated measures ν and μ).

Recall that if k is the $(d-1)$-dimensional Riesz kernel in $\mathbb{R}^d$ and μ has $(d-1)$-growth and is $(d-1)$ purely unrectifiable, that is $\mu(E) = 0$ for all $(d-1)$-rectifiable sets E, the principal values diverge μ almost everywhere and the weak convergence in $L^2(\mu)$ fails. On the other hand it is of interest that weak convergence in the sense of Theorem 1.1 holds as it holds for any s-dimensional antisymmetric kernel and any finite measure with s-growth.

2. Proof of Theorem 1.1

We first prove the following lemma about exponential growth of probability measures on compact intervals. It is motivated by a similar result proved in [SUZ]. Here

Leb stands for the Lebesgue measure on the real line and $|I|$ denotes the length of an interval $I \subset \mathbb{R}$.

Lemma 2.1. *For every integer $\lambda > 2$ the following holds. Let ν be a probability Borel measure on a compact interval $\Delta \subset \mathbb{R}$. Then for every interval $I \subset \Delta$ there exists a subset $I'(\lambda) \subset I$ such that $Leb(I'(\lambda)) > |I|(1 - 3(\lambda^{-1} + \lambda^{-2} + \dots))$ and for every $t \in I'(\lambda)$,*

$$\nu([t - \lambda^{3n}, t + \lambda^{3n}]) < \lambda^{-3n}$$

for all integers $n \geq 1$.

Proof. Let us partition the interval I into λ^2 subintervals J of length $|I|\lambda^{-2}$. Let B_1 be the family of all intervals J from this partition for which $\nu(J) < \lambda^{-1}$. Obviously, there are at most λ intervals in B_1^c. Thus

$$\# B_1 > \lambda^2 - \lambda = \lambda^2 \left(1 - \frac{\lambda}{\lambda^2}\right)$$

and

$$\text{Leb}\left(\bigcup\{J : J \in B_1\}\right) \geq |I| \left(1 - \frac{\lambda}{\lambda^2}\right) = |I| \left(1 - \frac{1}{\lambda}\right).$$

Next, each interval in B_1 is divided into λ^2 subintervals with disjoint interiors and of length $|I|\lambda^{-4}$, and we remove those subintervals for which $\nu(J) \geq \lambda^{-2}$. Denoting by B_2 the family of remaining intervals, we see that

$$\# B_2 \geq (\lambda^2)^2 \left(1 - \frac{\lambda}{\lambda^2}\right) - \lambda^2 = (\lambda^2)^2 \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

and

$$\text{Leb}\left(\bigcup\{J : J \in B_2\}\right) \geq |I| \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right).$$

Proceeding inductively, we partition the interval I into disjoint intervals of length $|I|\lambda^{-2n}$. Next, we define in the same way the family B_n. It is formed by the intervals J of this partition of n'th generation, which are contained in some interval of the family B_{n-1} and for which $\nu(J) < \lambda^{-n}$. Then

$$\text{Leb}\left(\bigcup\{J : J \in B_n\}\right) \geq \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2} - \dots - \frac{1}{\lambda^n}\right) |I|.$$

For any $t \in I$ let $J_n = J_n(t)$ be the interval of the n'th partition such that $t \in J_n$. Thus, for every $t \in \bigcap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, we have that $J_n(t) \in B_n$. Consequently, for all $t \in \bigcap_{n=1}^{\infty} \bigcup_{J \in B_n} J$, it holds that $\nu(J_n(t)) < \lambda^{-n}$ for all $n \geq 1$. Let now

$$C_n = \{t \in I : [t - |I|\lambda^{-3n}, t + |I|\lambda^{-3n}] \subset J_n(t)\}.$$

It is easy to see that $\text{Leb}(C_n^c) < 2|I|\lambda^{-n}$, and, therefore,

$$\text{Leb}\left(\bigcap_{n=1}^{\infty} C_n\right) > |I| \left(1 - 2\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \dots\right)\right).$$

Finally, setting

$$I' := \left(\bigcap_{n=1}^{\infty} C_n\right) \cap \left(\bigcap_{i=1}^{\infty} \bigcup_{J \in B_i} J\right)$$

completes the proof. $\square$

Proof of Theorem 1.1. We can assume that $\mu(X) \leq 1$. We define finite Borel measures on the unit interval for all $z \in \mathrm{spt}\mu$ by

$$\mu_z(F) = \mu\{x \in X : d(x,z) \in F\},\ F \subset [0,1].$$

Let $A_z = \cup_{\lambda>2} I'_z(\lambda)$ where $I'_z(\lambda)$ are the sets we obtain after we apply Lemma 2.1 to the measures μ_z. Then Lemma 2.1 implies that $\mu_z(A_z) = \mu_z([0,1])$. Let $G_z = \{r \in (0,1] : \ r \in A_z\}$ and

$$\mathcal{X}'_B = \left\{ \sum_{i=1}^{n} a_i \chi_{B(z_i,r_i)} : n \in \mathbb{N},\ a_i \in \mathbb{R},\ z_i \in \mathrm{spt}\mu,\ r_i \in G_{z_i} \right\}.$$

Then $\mathcal{X}'_B$ is dense in $L^2(\mu)$.

Let $f, g \in \mathcal{X}'_B$ such that

$$f = \sum_{i}^{n} a_i \chi_{B_i} \text{ and } g = \sum_{j}^{m} b_j \chi_{S_j},$$

where $a_i, b_j \in \mathbb{R}$ and B_i, S_j are closed balls. Then for $0 < \delta < \varepsilon$,

$$\int T^k_{\mu,\varepsilon} f(x) g(x) d\mu(x) - \int T^k_{\mu,\delta} f(x) g(x) d\mu(x) = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i b_j \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j} \int_{B_i}} k(x,y) d\mu(y) d\mu(x).$$

Furthermore,

$$\begin{aligned}
&\left| \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j} \int_{B_i}} k(x,y) d\mu(y) d\mu(x) \right| \\
&\leq \left| \underset{\delta<d(x,y)<\varepsilon}{\int_{B_i \cap S_j} \int_{B_i \cap S_j}} k(x,y) d\mu(y) d\mu(x) \right| + \left| \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j \setminus B_i} \int_{B_i \cap S_j}} k(x,y) d\mu(y) d\mu(x) \right| \\
&\quad + \left| \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j \setminus B_i} \int_{B_i \setminus S_j}} k(x,y) d\mu(y) d\mu(x) \right| + \left| \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j \cap B_i} \int_{B_i \setminus S_j}} k(x,y) d\mu(y) d\mu(x) \right| \\
&\leq \underset{\delta<d(x,y)<\varepsilon}{\int_{B_i} \int_{B_i^c}} |k(x,y)|\, d\mu(y) d\mu(x) + 2 \underset{\delta<d(x,y)<\varepsilon}{\int_{S_j} \int_{S_j^c}} |k(x,y)|\, d\mu(y) d\mu(x).
\end{aligned}$$

The last inequality follows because by antisymmetry and Fubini's theorem

$$\int_{\substack{B_i \cap S_j \\ \delta < d(x,y) < \varepsilon}} \int_{B_i \cap S_j} k(x,y) d\mu(y) d\mu(x) = 0.$$

Therefore it is enough to show that for any "good" ball $B = B(z,r)$ with $z \in \operatorname{spt}\mu$ and $r \in G_z$

$$\lim_{\substack{0<\delta<\varepsilon \\ \varepsilon \to 0}} \int_{\substack{B \\ \delta < d(x,y) < \varepsilon}} \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) = 0,$$

which will follow by the monotone convergence theorem if we show that

$$\int_B \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty. \tag{2.1}$$

Since $B = B(z,r)$ and $r \in G_z$ Lemma 2.1 implies that $\mu(\partial B) = 0$ hence it is enough to show that

$$\int_{B^o} \int_{B^c} |k(x,y)| d\mu(y) d\mu(x) < \infty$$

where B^o stands for the interior of B. For any $x \in B^o$ let $n(x) > 0$ such that

$$2^{n(x)} d(x, \partial B) = 3$$

and $N(x) =$ integer part of $n(x) + 1$. Therefore, since $\operatorname{diam}(B) \leq 1$,

$$B(x,2) \setminus B \subset \cup_{i=1}^{N(x)} B(x, 2^i d(x, \partial B)) \setminus B(x, 2^{i-1} d(x, \partial B)).$$

Hence for all $x \in B^o$

$$\begin{aligned}
\int_{B(x,2) \setminus B} |k(x,y)| d\mu(y) &\leq \int_{B(x,2) \setminus B} d(x,y)^{-s} d\mu(y) \\
&= \sum_{i=1}^{N(x)} \int_{B(x, 2^i d(x,\partial B)) \setminus B(x, 2^{i-1} d(x,\partial B))} d(x,y)^{-s} d\mu(y) \\
&\leq \sum_{i=1}^{N(x)} \mu(B(x, 2^i d(x, \partial B))(2^{i-1} d(x, \partial B))^{-s} d\mu(y) \\
&\lesssim N(x) \lesssim |\log d(x, \partial B)|,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B^c} |k(x,y)| d\mu(y) &\lesssim \int_{B(x,2)^c} d(x,y)^{-s} d\mu(y) + |\log d(x, \partial B)| \\
&\lesssim 1 + |\log d(x, \partial B)|.
\end{aligned}$$

Since $r \in G_z$ there exists some $\lambda \in \mathbb{N}$ such that $r \in I'_z(\lambda)$. We write,

$$\int_{B(z,r)^o} |\log d(x,\partial B)| d\mu(x) = \int_{B(z,r-\lambda^{-3})^o} |\log d(x,\partial B)| d\mu(x) + \sum_{n=1}^{\infty} \int_{\{x : r-\lambda^{-3n} \leq d(z,x) < r-\lambda^{-3(n+1)}\}} |\log d(x,\partial B)| d\mu(x)$$

Notice that by Lemma 2.1

$$\mu(\{x : r - \lambda^{-3n} \leq d(z,x) < r - \lambda^{-3(n+1)}\}) = \mu_z([r - \lambda^{-3n}, r - \lambda^{-3(n+1)})) \leq \mu_z([r - \lambda^{-3n}, r + \lambda^{-3n})) \leq \lambda^{-n}.$$

Therefore,

$$\int_{B(z,r)^o} |\log d(x,\partial B)| d\mu(x) \lesssim 3\log(\lambda)(r - \lambda^{-3})^s + \sum_{i=1}^{n} \lambda^{-n} |\log(\lambda^{-3(n+1)})| < \infty$$

and this completes the proof of Theorem 1.1. □

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